

POSITIVITY OF ANTI-CANONICAL DIVISORS AND F -PURITY OF FIBERS

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ABSTRACT. In this paper, we prove that given a flat generically smooth morphism between smooth projective varieties with F -pure closed fibers, if the source space is Fano, weak Fano or a variety with the nef anti-canonical divisor, then so is the target space. We also show that relative anti-canonical divisors of generically smooth surjective nonconstant morphisms are not nef and big in arbitrary characteristic.

1. INTRODUCTION

Let $f : X \rightarrow Y$ be a morphism between smooth projective varieties over an algebraically closed field k . Kollár, Miyaoka and Mori [14, Corollary 2.9] proved that, under the assumption that f is smooth, if X is a Fano variety, that is, $-K_X$ is ample, then so is Y . It follows from an analogous argument that, under the same assumption, if $-K_X$ is nef, then so is $-K_Y$ (cf. [17], [8, Theorem 1.1] and [5, Corollary 3.15 (a)]). Based on these results, Yasutake asked: “what positivity condition is passed from $-K_X$ to $-K_Y$?” Some answers to this question are known in characteristic of k is zero. Fujino and Gongyo [7, Theorem 1.1] proved that, under the assumption that f is smooth, if X is a weak Fano variety, that is, $-K_X$ is nef and big, then so is Y . Birkar and Chen [1, Theorem 1.1] showed that, under the same assumption, if $-K_X$ is semi-ample, then so is $-K_Y$. Furthermore, similar but weaker results holds even if f is not smooth (but the characteristic of k is still zero). For example, a result of Prokhorov and Shokurov [22, Lemma 2.8] (cf. [7, Corollary 3.3]) implies that if $-K_X$ is nef and big, then $-K_Y$ is big. Chen and Zhang [2, Main theorem] also proved that if $-K_X$ is nef, then $-K_Y$ is pseudo-effective.

In contrast, little was known about the positive characteristic case. In this paper, assuming that the geometric generic fiber has only F -pure or strongly F -regular singularities, we prove that (generalizations of) the statements in the above holds in positive characteristic, except for the one on semi-ampleness. F -purity and strong F -regularity are mild singularities defined in terms of Frobenius splitting properties (Definitions 2.1 and 2.2), which have a close connection to log canonical and Kawamata log terminal singularities respectively.

Suppose that the characteristic of k is equal to $p > 0$. Let Δ be an effective \mathbb{Q} -divisor on X which is \mathbb{Q} -Cartier with index m , and let D be a \mathbb{Q} -divisor on Y which is \mathbb{Q} -Cartier with index n . Let $\bar{\eta}$ be the geometric generic point of Y . Then our main theorem is stated as follows.

Theorem 1.1 (Theorem 4.2). *Let S be a subset of Y such that the following conditions holds for every $y \in S$:*

- (i) $\dim X_y = \dim X - \dim Y$.
- (ii) *The support of Δ does not contain any irreducible component of X_y .*
- (iii) *$(X_{\bar{y}}, \Delta_{\bar{y}})$ is F -pure, where \bar{y} is the algebraic closure of y .*

*If $p \nmid m$ and if $-(K_X + \Delta + f^*D)$ is nef, then $\mathcal{O}_Y(-n(K_Y + D))$ is weakly positive over an open subset of Y containing S .*

Here, weak positivity over $Y_0 \subseteq Y$ of line bundles on Y is a property which can be viewed as a generalization of nefness (Definition 3.1), and is equivalent to nefness if $Y_0 = Y$ (Remark 3.3). In Section 4, Theorem 4.2 is proved in a more general setting.

The following two theorems are corollaries of Theorem 1.1.

Theorem 1.2 (Corollary 4.4). *Assume that f is flat, the support of Δ does not contain any component of any fiber, and that (X_y, Δ_y) is F -pure for every closed point $y \in Y$.*

- (1) *If $p \nmid m$ and if $-(K_X + \Delta + f^*D)$ is nef, then so is $-(K_Y + D)$.*
- (2) *If $-(K_X + \Delta + f^*D)$ is ample, then so is $-(K_Y + D)$.*

Theorem 1.3 (Corollary 4.5). *Assume that $(X_{\bar{\eta}}, \Delta_{\bar{\eta}})$ is F -pure.*

- (1) *If $p \nmid m$ and if $-(K_X + \Delta + f^*D)$ is nef, then $-(K_Y + D)$ is pseudo-effective.*
- (2) *If $-(K_X + \Delta + f^*D)$ is ample, then $-(K_Y + D)$ is big.*
- (3) *If $(X_{\bar{\eta}}, \Delta_{\bar{\eta}})$ is strongly F -regular and if $-(K_X + \Delta + f^*D)$ is nef and big, then $-(K_Y + D)$ is big.*

Theorem 1.2 is a generalization of [14, Corollary 2.9] and [5, Corollary 3.15] in positive characteristic. We can also recover [14, Corollary 2.9] in characteristic zero from Theorem 1.2 by standard reduction to characteristic p techniques. Our proof relies on a study of the positivity of direct image sheaves for f in terms of the Grothendieck trace of the relative Frobenius morphism. This is completely different from the proof of Kollár, Miyaoka and Mori which relies on a detailed study of rational curves on varieties. Theorem 1.3 should be compared with [22, Lemma 2.8] and [2, Main Theorem].

The following two theorems are direct consequences of Theorems 1.2 and 1.3.

Theorem 1.4 (Corollary 4.6). *Assume that $(X_{\bar{\eta}}, \Delta_{\bar{\eta}})$ is F -pure. If $p \nmid m$ and if $K_X + \Delta \sim_{\mathbb{Q}} f^*(K_Y + L)$ for some \mathbb{Q} -divisor L on Y , then L is pseudo-effective.*

Theorem 1.5 (Corollary 4.9). *Assume that f is flat, that every closed fiber is F -pure, and that the geometric generic fiber is strongly F -regular. If X is a weak Fano variety, that is, $-K_X$ is nef and big, then so is Y .*

Theorem 1.5 is a positive characteristic counterpart of [7, Theorem 1.1].

For another application of Theorem 1.1, we return to the situation where k is of arbitrary characteristic. Suppose that $f : X \rightarrow Y$ is a generically smooth surjective morphism between smooth projective varieties over an algebraically closed field of arbitrary characteristic and in addition that the dimension of Y is positive.

Theorem 1.6 (Corollary 4.10 and Theorem 5.4). *$-K_{X/Y}$ is not nef and big.*

Theorem 1.7 (Corollary 4.11 and Theorem 5.5). *Assume that $\omega_{X/\eta}^{-m}$ is globally generated for an integer $m > 0$. Then $f_*\omega_{X/Y}^{-m}$ is not big in the sense of Definition 3.1.*

In both theorem, the characteristic zero case is proved by reduction to positive characteristic. Theorem 1.6 improves a result of Kollár, Miyaoka and Mori [14, Corollary 2.8] which states that $-K_{X/Y}$ is not ample. Theorem 1.7 includes a result of Miyaoka [17, COROLLARY 2'] which states that if $\omega_{X/Y}^{-1}$ is f -ample and if $\omega_{X/Y}^{-m}$ is f -free for an integer $m > 0$, then $f_*\omega_{X/Y}^{-m}$ is not an ample vector bundle.

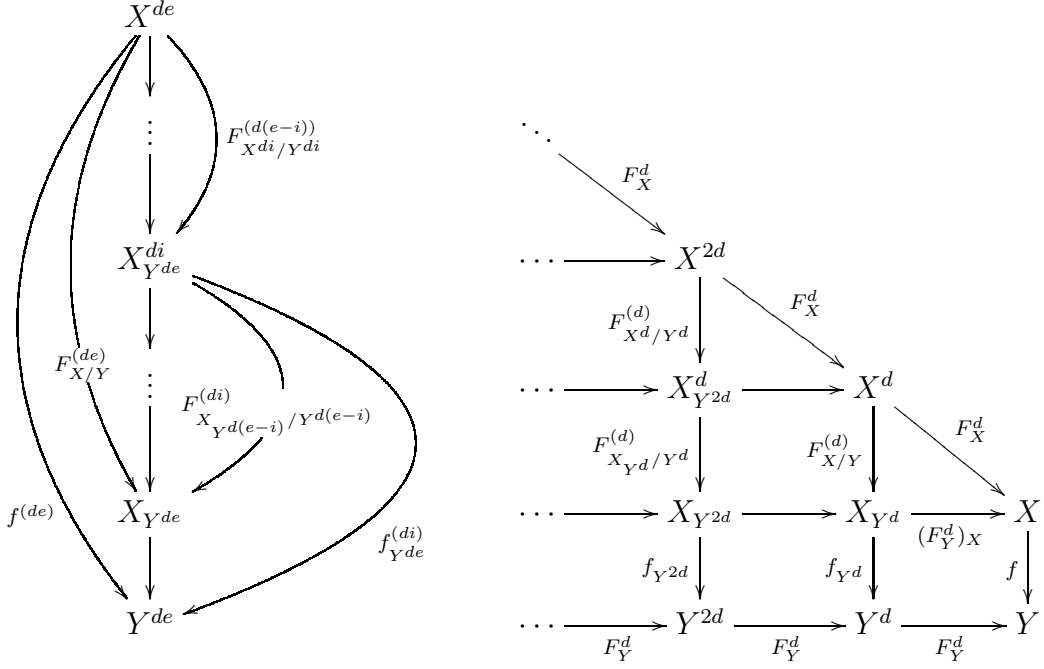
1.1. Notation. In this paper, we work over an algebraically closed field k . A k -scheme is a separated scheme of finite type over k . A *variety* means an integral k -scheme. Let $\varphi : S \rightarrow T$ be a morphism of schemes and let T' be a T -scheme. Then we denote by $S_{T'}$ and $\varphi_{T'} : S_{T'} \rightarrow T'$ respectively the fiber product $S \times_T T'$ and its second projection. For a Cartier or \mathbb{Q} -Cartier divisor D on S (resp. an \mathcal{O}_S -module \mathcal{G}), the pullback of D (resp. \mathcal{G}) to $S_{T'}$ is denoted by $D_{T'}$ (resp. $\mathcal{G}_{T'}$) if it is well-defined. Similarly, for a homomorphism of \mathcal{O}_S -modules $\alpha : \mathcal{F} \rightarrow \mathcal{G}$, the pullback of α to $S_{T'}$ is denoted by $\alpha_{T'} : \mathcal{F}_{T'} \rightarrow \mathcal{G}_{T'}$. For a scheme X of positive characteristic, $F_X : X \rightarrow X$ is the absolute Frobenius morphism. We often denote the source of F_X^e by X^e . Let $f : X \rightarrow Y$ be a morphism between schemes of positive characteristic. The same morphism is denoted by $f^{(e)} : X^e \rightarrow Y^e$ when we regard X (resp. Y) as X^e (resp. Y^e). We define the e -th relative Frobenius morphism of f to be the morphism $F_{X/Y}^{(e)} := (F_X^e, f^{(e)}) : X^e \rightarrow X \times_Y Y^e =: X_{Y^e}$. For a prime $p \in \mathbb{Z}$, $\mathbb{Z}_{(p)}$ denotes the localization of \mathbb{Z} at $(p) = p\mathbb{Z}$.

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2. TRACE MAPS OF RELATIVE FROBENIUS MORPHISMS

In this section, we work over an algebraically closed field of characteristic $p > 0$. Let $f : X \rightarrow Y$ be a morphism from a pure dimensional Gorenstein k -scheme X to a smooth variety Y . Let K_X be a Cartier divisor such that $\mathcal{O}_X(K_X)$ is isomorphic to the dualizing sheaf ω_X of X . Set $K_{X/Y} := K_X - f^*K_Y$. For each $d, e > 0$ we

define some notation by the following commutative diagram.



Since F_Y is flat, every horizontal morphism in the diagram is a Gorenstein morphism, and thus every object in the diagram is a pure dimensional Gorenstein k -scheme. We denote by $\mathrm{Tr}_{F_{X/Y}^{(1)}} : F_{X/Y}^{(1)} \omega_{X^1} \rightarrow \omega_{X_{Y^1}}$ the morphism obtained by applying the functor $\mathcal{H}om_{\mathcal{O}_{X_{Y^1}}}(_, \omega_{X_{Y^1}})$ to the natural morphism $F_{X/Y}^{(1)\#} : \mathcal{O}_{X_{Y^1}} \rightarrow F_{X/Y}^{(1)} \mathcal{O}_X$. For each $e > 0$ we define

$$\begin{aligned} \phi_{X/Y}^{(1)} &:= \mathrm{Tr}_{F_{X/Y}^{(1)}} \otimes \mathcal{O}_{X_{Y^1}}(-K_{X_{Y^1}}) : F_{X/Y}^{(1)} \mathcal{O}_{X^1}((1-p)K_{X^1/Y^1}) \rightarrow \mathcal{O}_{X_{Y^1}}, \quad \text{and} \\ \phi_{X/Y}^{(e+1)} &:= \left(\phi_{X/Y}^{(e)} \right)_{Y^{e+1}} \circ F_{X_{Y^1}/Y^1}^{(e)} \left(\phi_{X^e/Y^e}^{(1)} \otimes \mathcal{O}_{X_{Y^{e+1}}}((1-p^e)K_{X_{Y^{e+1}}/Y^{e+1}}) \right) \\ &\quad : F_{X/Y}^{(e+1)} \mathcal{O}_X((1-p^{e+1})K_{X^{e+1}/Y^{e+1}}) \rightarrow \mathcal{O}_{X_{Y^{e+1}}}. \end{aligned}$$

Let E be an effective Cartier divisor on X , let $a > 0$ be an integer not divisible by p , and let $d > 0$ be the smallest integer satisfying $a|(p^d - 1)$. For each $e > 0$ we define

$$\begin{aligned} \mathcal{L}_{(X/Y, E/a)}^{(de)} &:= \mathcal{O}_{X^{de}}((1-p^{de})a^{-1}(aK_{X^{de}/Y^{de}} + E)) \subseteq \mathcal{O}_{X^{de}}((1-p^{de})K_{X^{de}/Y^{de}}), \\ \phi_{(X/Y, E/a)}^{(d)} &: F_{X/Y}^{(d)} \mathcal{L}_{(X/Y, E/a)}^{(d)} \rightarrow F_{X/Y}^{(d)} \mathcal{O}_{X^d}((1-p^d)K_{X^d/Y^d}) \xrightarrow{\phi_{X/Y}^{(d)}} \mathcal{O}_{X_{Y^d}}, \quad \text{and} \\ \phi_{(X/Y, E/a)}^{(d(e+1))} &:= \left(\phi_{(X/Y, E/a)}^{(de)} \right)_{Y^{de}} \circ F_{X_{Y^d}/Y^d}^{(de)} \left(\phi_{(X^{de}/Y^{de}, E^{de}/a)}^{(d)} \otimes (\mathcal{L}_{(X/Y, E/a)}^{(de)})_{Y^{d(e+1)}} \right) \\ &\quad : F_{X/Y}^{(d(e+1))} \mathcal{L}_{(X/Y, E/a)}^{(d(e+1))} \rightarrow \mathcal{O}_{X_{Y^{d(e+1)}}}. \end{aligned}$$

Let $f : X \rightarrow Y$ be a morphism from a pure dimensional Gorenstein k -schemes X to a smooth variety Y . Assume that X satisfies S_2 and G_1 . Let E be an effective AC divisor on X (cf. [16]), let $a > 0$ be an integer not divisible by p , and let $d > 0$

be the smallest integer satisfying $a|(p^d - 1)$. Then for each $e > 0$ we define

$$\begin{aligned}\mathcal{L}_{(X/Y, E/a)}^{(de)} &:= \iota_{Y^{de}*} \mathcal{L}_{(U/Y, E|_U/a)}^{(de)}, \text{ and} \\ \phi_{(X/Y, E/a)}^{(de)} &:= \iota_{Y^{de}*} (\phi_{(U/Y, E|_U/a)}^{(de)}) : F_{X/Y}^{(de)} \mathcal{L}_{(X/Y, E/a)}^{(de)} \rightarrow \mathcal{O}_{X_{Y^{de}}},\end{aligned}$$

where $\iota : U \hookrightarrow X$ is a Gorenstein open subset of X such that $\text{codim } X \setminus U \geq 2$ and that $E|_U$ is a Cartier divisor on U . If X is normal, then for each $e > 0$ such that $E := (p^e - 1)\Delta$ is integral we denote $\mathcal{L}_{(X/Y, E/p^{e-1})}^{(e)}$ and $\phi_{(X/Y, E/p^{e-1})}^{(e)}$ respectively by $\mathcal{L}_{(X/Y, \Delta)}^{(e)}$ and $\phi_{(X/Y, \Delta)}^{(e)}$.

Next we introduce singularities of pairs defined by the Grothendieck trace of the Frobenius morphism. $\text{WSh}(X)$ denotes the set of AC-divisors on a k -scheme X of pure dimension satisfying S_2 and G_1 . A $\mathbb{Z}_{(p)}$ -AC divisor on X is an element of $\text{WSh}(X) \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$.

Definition 2.1. Let X be a k -scheme of pure dimension satisfying S_2 and G_1 , and let Δ be an effective \mathbb{Q} -AC divisor on X . Set $Y := \text{Spec } k$. The pair (X, Δ) is said to be F -pure if for every $e > 0$ and for every effective AC divisor E with $E \leq (p^e - 1)\Delta$ the morphism

$$\phi_{(X/Y, E/(p^e-1))}^{(e)} : F_{X/Y}^{(e)} \mathcal{O}_X((1 - p^e)K_X - E) \rightarrow \mathcal{O}_{X_{Y^e}}$$

is surjective. We simply say that X is F -pure if $(X, 0)$ is F -pure.

Definition 2.2. Let X be a normal variety, and let Δ be an effective \mathbb{Q} -divisor on X . Set $Y := \text{Spec } k$. The pair (X, Δ) is said to be *strongly F -regular* if for every effective divisor D , there exists an $e > 0$ such that

$$\phi_{(X/Y, \lceil (p^e-1)\Delta \rceil + D/p^{e-1})}^{(e)} : F_{X/Y}^{(e)} \mathcal{O}_X(\lfloor (1 - p^e)(K_X + \Delta) \rfloor - D) \rightarrow \mathcal{O}_{X_{Y^e}}$$

is surjective. Here $\lceil \Delta \rceil$ (resp. $\lfloor \Delta \rfloor$) denotes the round up (resp. down) of Δ . We simply say that X is *strongly F -regular* if $(X, 0)$ is *strongly F -regular*.

Remark 2.3. Let (X, Δ) be a strongly F -regular pair, and let Δ' be an effective \mathbb{Q} -divisor on X . Then there exists an $0 < \varepsilon \in \mathbb{Q}$ such that $(X, \Delta + \varepsilon\Delta')$ is again strongly F -regular.

Lemma 2.4. Let $f : X \rightarrow Y$ be a flat projective morphism from a pure dimensional Gorenstein k -scheme X to a smooth variety Y . Let E be an effective Cartier divisor on X whose support does not contain any component of any fiber, and let $a > 0$ be an integer not divisible by p . Set $\Delta := E \otimes a^{-1}$.

- (1) [21, Corollary 3.31] The set $Y_0 := \{y \in Y \mid (X_{\bar{y}}, \Delta_{\bar{y}}) \text{ is } F\text{-pure}\}$ is an open subset of Y , where \bar{y} is the algebraic closure of y .
- (2) Let A be a Cartier divisor on X such that A_{Y_0} is f_{Y_0} -ample. Then there exists an $m_0 > 0$ such that

$$\begin{aligned}f_{Y^e*}(\phi_{(X/Y, E/a)}^{(e)} \otimes \mathcal{O}_{Y^e}(m_0 A_{Y^e} + N_{Y^e})) : \\ f_{X^e}^{(e)} \mathcal{O}_{X^e}((1 - p^e)a^{-1}(aK_{X^e/Y^e} + E) + p^e(m_0 A + N)) \rightarrow f_{Y^e*} \mathcal{O}_{X^e}(m_0 A_{Y^e} + N_{Y^e})\end{aligned}$$

is surjective over Y_0 for every Cartier divisor N on X whose restriction N_{Y_0} to Y_0 is f_{Y_0} -nef and for every $e > 0$ with $a|(p^e - 1)$.

Proof. We first note that f is a Gorenstein morphism, and hence every fiber is Gorenstein. If $(X_{\bar{y}}, \Delta_{\bar{y}})$ is F -pure for some $y \in Y$, then $\phi_{(X/Y, E/a)}^{(e)}$ is surjective over $f_{Y^{de}}^{-1}(y)$ by [21, Lemma 2.17]. Thus there exists an open subset $V \subseteq Y$ such that $\phi_{(X/Y, E/a)}^{(e)}$ is surjective over $f^{-1}(V)$. Applying [21, Lemma 2.17] again, we get $V \subseteq Y_0$, and so (1) is proved. To prove (2), we may assume that $Y_0 = Y$. Let $d > 0$ be the minimum integer such that $a|(p^d - 1)$. Since $\phi_{(X/Y, E/a)}^{(d)}$ is surjective, there exists an integer $m_0 > 0$ such that $f_{Y^d}^*(\phi_{(X/Y, E/a)}^{(d)} \otimes \mathcal{O}_{X_{Y^d}}((m_0 A + N)_{Y^d}))$ is surjective for every f -nef Cartier divisor N on X because of Keeler's relative Fujita vanishing [12, Theorem 1.5]. We may assume that $m_0 A - (K_{X/Y} + \Delta)$ is f -nef. Then by the definition, we have

$$\begin{aligned} & f_{Y^{d(e+1)}}^*(\phi_{(X/Y, E/a)}^{(d(e+1))} \otimes \mathcal{O}_{X_{Y^{d(e+1)}}}((m_0 A + N)_{Y^{d(e+1)}})) \\ & \cong F_Y^{d*} \left(f_{Y^{de}}^*(\phi_{(X/Y, E/a)}^{(de)} \otimes \mathcal{O}_{X_{Y^{de}}}((m_0 A + N)_{Y^{de}})) \right) \\ & \circ f_{Y^{d(e+1)}}^{(de)*} \left(\phi_{(X^{de}/Y^{de}, E^{de}/a)}^{(d)} \otimes (\mathcal{L}_{(X/Y, E/a)}^{(de)}(p^{de}(m_0 A + N)))_{Y^{d(e+1)}} \right). \end{aligned}$$

Here, $\mathcal{L}_{(X/Y, E/a)}^{(de)}(p^{de}(m_0 A + N))$ is isomorphic to

$$\mathcal{O}_{X^{de}}(m_0 A + (p^{de} - 1)a^{-1}(am_0 A - (aK_{X^{de}/Y^{de}} + E)) + p^{de}N).$$

Thus the assertion is satisfied by the hypothesis of the induction. \square

3. WEAK POSITIVITY OVER S AND A NUMERICAL INVARIANT

In this section, we define an invariant of coherent sheaves on normal varieties of positive characteristic which measures positivity. This will play an important role in the proof of the main theorem.

We first recall some definitions of the positivity of coherent sheaves on normal varieties over an algebraically closed field k of arbitrary characteristic.

Definition 3.1. Let Y be a quasi-projective normal variety, let \mathcal{G} be a coherent sheaf on Y , and let H be an ample Cartier divisor. Let S be a non-empty subset of Y .

- (i) \mathcal{G} is said to be *globally generated over S* if the natural morphism $H^0(Y, \mathcal{G}) \otimes_k \mathcal{O}_Y \rightarrow \mathcal{G}$ is surjective over S .
- (ii) \mathcal{G} is said to be *weakly positive over S* if for every $a > 0$, there exists $b > 0$ such that $(S^{ab}\mathcal{G})^{**} \otimes \mathcal{O}_Y(bH)$ is globally generated over S . Here $S^{ab}(_)$ and $(_)^{**}$ denote the ab -th symmetric product and the double dual respectively.
- (iii) \mathcal{G} is said to be *big over S* if there exists an $a > 0$ such that $(S^a\mathcal{G})(-H)$ is weakly positive over S .

We say simply that \mathcal{G} is weakly positive over y when $S = \{y\}$ for a point $y \in Y$. \mathcal{G} is said to be *weakly positive* (resp. *big*) if \mathcal{G} is weakly positive (resp. big) over the generic point η of Y .

Remark 3.2. The notion of weak positivity is first introduced by Viehweg as a generalization of nefness of vector bundles, when S is an open subset [25]. In [13], [20] and [6], (resp. [18]), this notion is also defined in the case when $S = \{\eta\}$ (resp. $S = \{y\}$ for a point $y \in Y$).

Remark 3.3. (1) Definition 3.1 is independent of the choice of H (cf. [25, Lemma 2.14]).

(2) Let $Y_0 \subseteq Y$ be an open subset such that $S \subseteq Y_0$ and $\text{codim}(Y_0 \setminus Y) \geq 2$. Then \mathcal{G} is weakly positive over S if and only if so is $\mathcal{G}|_{Y_0}$.

(3) We set $t'_S(\mathcal{G}, H) := \sup T'_S(\mathcal{G}, H)$, where

$$T'_S(\mathcal{G}, H) := \left\{ \varepsilon \in \mathbb{Q} \left| \begin{array}{l} \text{there exist } a, b \in \mathbb{Z} \text{ such that} \\ \varepsilon = a/b, b > 0, \text{ and } (S^b \mathcal{G})^{**}(-aH) \text{ is} \\ \text{globally generated over } S. \end{array} \right. \right\}.$$

It is clear that $T'_S(\mathcal{G}, H)$ is equal to $\mathbb{Q} \cap (-\infty, t'_S(\mathcal{G}, H))$ or $\mathbb{Q} \cap (-\infty, t'_S(\mathcal{G}, H)]$. So \mathcal{G} is weakly positive (resp. big) over S if and only if $t'_S(\mathcal{G}, H) \geq 0$ (resp. > 0).

(4) Assume that \mathcal{G} is a vector bundle and that Y is projective. Then \mathcal{G} is weakly positive (resp. big) over Y if and only if \mathcal{G} is nef (resp. ample).

(5) Assume that \mathcal{G} is a line bundle and that Y is projective. Then \mathcal{G} is weakly positive (resp. big) over the generic point η of Y if and only if \mathcal{G} is pseudo-effective (resp. big).

From now on, we suppose that the characteristic of the base field k is positive.

Definition 3.4. Let Y, \mathcal{G}, S be as in Definition 3.1, and assume that the characteristic of k is $p > 0$. Let D be a \mathbb{Q} -Cartier divisor. Then we define

$$T_S(\mathcal{G}, D) := \left\{ \varepsilon \in \mathbb{Q} \left| \begin{array}{l} \text{there exists an } e > 0 \text{ such that} \\ p^e \varepsilon D \text{ is Cartier and } (F_Y^e \mathcal{G})(-p^e \varepsilon D) \text{ is} \\ \text{globally generated over } S. \end{array} \right. \right\}, \text{ and} \\ t_S(\mathcal{G}, D) := \sup T_S(\mathcal{G}, D) \in \mathbb{R} \cup \{-\infty, +\infty\}.$$

Lemma 3.5. *Under the same assumption as the above, let \mathcal{F} be a coherent sheaf on Y .*

(1) *If there exists a morphism $\mathcal{F} \rightarrow \mathcal{G}$ such that surjective over S , then $t_S(\mathcal{F}, D) \leq t_S(\mathcal{G}, D)$.*

(2) *Assume that $\{t_S(\mathcal{F}, D), t_S(\mathcal{G}, D)\} \neq \{-\infty, +\infty\}$. Then*

$$t_S(\mathcal{F}, D) + t_S(\mathcal{G}, D) \leq t_S(\mathcal{F} \otimes \mathcal{G}, D).$$

(3) *For each $e > 0$, $t_S(F_Y^e \mathcal{G}, D) = p^e t_S(\mathcal{G}, D)$.*

(4) *If \mathcal{G}_y is free \mathcal{O}_y -module for every $y \in S$ and if $t_S(\mathcal{G}, D) = +\infty$, then $\mathcal{O}_Y(-bD)$ is weakly positive over S for some $b > 0$ such that bD is Cartier.*

Proof. (1)–(3) follow directly from the definition. We prove (4). There is a natural morphism $\mathcal{G}^* \otimes \mathcal{G} \rightarrow \mathcal{O}_Y$, which is surjective over S by the assumption. Furthermore, there is an ample Cartier divisor such that $\mathcal{G}^*(H)$ is globally generated, and thus we have a surjective morphism $\mathcal{O}_Y^{\oplus h} \rightarrow \mathcal{G}^*(H)$ for some $h > 0$. Hence we get a morphism

$$\mathcal{G}^{\oplus h} \rightarrow \mathcal{G}^* \otimes \mathcal{G}(H) \rightarrow \mathcal{O}_Y(H)$$

which is surjective over S . By (1) we have $t_S(\mathcal{O}_Y(H), D) = +\infty$, and thus there exists a sequence $\{\varepsilon_n\}_{n \geq 1}$ of elements of $T_S(\mathcal{O}_Y(H), D) \setminus \{0\}$ such that $\varepsilon_n \xrightarrow{n \rightarrow +\infty} +\infty$. By definition, for every $n \geq 1$ there exists $e \geq 1$

$$(F_Y^e \mathcal{O}_Y(H))(-p^e \varepsilon_n D) \cong \mathcal{O}_Y(p^e \varepsilon_n (-D + \varepsilon_n^{-1} H))$$

is globally generated over S . Using the notation of 3.3, we see that $-b/\varepsilon_n \leq t'_S(\mathcal{O}_Y(-bD), H)$, and so $0 \leq t'_S(\mathcal{O}_Y(-bD), H)$. This implies $\mathcal{O}_Y(-bD)$ is weakly positive over S . \square

4. MAIN THEOREM

In this section, we work over an algebraically closed field k of characteristic $p > 0$. The purpose of this section is to prove Theorem 4.2. First we show the following lemma.

Lemma 4.1. *Let $f : X \rightarrow Y$ be a surjective morphism from a projective k -scheme X to a projective variety Y , let A be an ample Cartier divisor on X , and let \mathcal{G} be a coherent sheaf on X . Then there exists an $m_0 > 0$ such that for every nef Cartier divisor N , $f_*(\mathcal{G}(m_0A + N))$ is globally generated over Y and locally free over Y_0 , where $Y_0 \subseteq Y$ is an open subset such that \mathcal{G}_{Y_0} is flat over Y_0 .*

Proof. Let H be an ample and free Cartier divisor on Y , and let $m_1 > 0$ be an integer such that $m_1A - \dim Y f^*H$ is nef. By the Fujita vanishing and Keeler's relative Fujita vanishing, there exists an $m_2 > 0$ such that for every nef Cartier divisor N on X and for each $i > 0$,

$$H^i(X, \mathcal{G}(m_2A + N)) = 0 \quad \text{and} \quad R^i f_*(\mathcal{G}(m_2A + N)) = 0.$$

Thus, by the Leray spectral sequence, we have

$$H^i(Y, (f_*(\mathcal{G}((m_1+m_2)A+N)))(-iH)) = H^i(Y, f_*(\mathcal{G}(m_2A+(m_1A-if^*H)+N))) = 0$$

for each $i > 0$. We set $m_0 = m_1 + m_2$. Then the global generation follows from the Castelnuovo-Mumford regularity [15, Theorem 1.8.5]. Furthermore we have

$$H^i(X_y, \mathcal{G}_y(m_0A_y + N_y)) = 0$$

for every $y \in Y_0$ and for each $i > 0$ [11, Theorem 12.11], and hence we see that $f_*\mathcal{G}(m_0A + N)$ is locally free over Y_0 [11, Theorem 9.9, Corollary 12.9]. \square

Let $f : X \rightarrow Y$ be a surjective projective morphism from a pure dimensional quasi-projective k -scheme X satisfying S_2 and G_1 to a normal quasi-projective variety Y , let E be an effective AC-divisor on X and let $a > 0$ be an integer. Set $\Delta := E \otimes a^{-1}$. Let D be a \mathbb{Q} -Cartier divisor on Y and let $b > 0$ be an integer such that bD is Cartier. Let S be a subset of Y such that the following conditions holds for every $y \in S$:

- (i) Y is smooth around y and X is Gorenstein around X_y .
- (ii) $\dim X_y = \dim X - \dim Y$.
- (iii) E is Cartier around X_y and its support does not contain any irreducible component of X_y .
- (iv) $(X_{\bar{y}}, \Delta_{\bar{y}})$ is F -pure, where \bar{y} is the algebraic closure of y .

The following two theorems are proved by the same argument.

Theorem 4.2. *Assume that X is projective and $K_X + \Delta$ is \mathbb{Q} -Cartier.*

- (1) *If $p \nmid a$ and $-(K_X + \Delta + f^*D)$ is nef, then $\mathcal{O}_Y(-b(K_Y + D))$ is weakly positive over an open subset of Y containing S .*
- (2) *If K_X is \mathbb{Q} -Cartier and $-(K_X + \Delta + f^*D)$ is ample, then $\mathcal{O}_Y(-b(K_Y + D))$ is big over an open subset of Y containing S .*

Theorem 4.3. *Assume that $\mathcal{O}_X(-b(aK_X + E))_y$ is free for every $y \in S$. If $p \nmid a$ and $f_*\mathcal{O}_X(-b(aK_X + E + af^*D))$ is globally generated over S , then $\mathcal{O}_Y(-b(K_Y + D))$ is weakly positive over S .*

Proof of Theorem 4.2. First we prove (1). By Lemma 2.4 (1), there exists an open subset $Y_1 \subseteq Y$ containing S such that (i)–(iv) above are satisfied for every $y \in Y_1$. Thus we may assume that S is open. We show that $\mathcal{O}_Y(-b(K_Y + L))$ is weakly positive over S . Replacing Y by its appropriate open subset, by Remark 3.3, we may assume that Y is smooth and f is flat. Let $d > 0$ be an integer divisible by a such that dD and $da^{-1}(aK_X + E)$ is Cartier. Let A be an ample and free Cartier divisor on X such that for every nef Cartier divisor N on X and for each $0 \leq r < d$ with $a|r$,

$$f_*((f^*\omega_Y^{\otimes r})(-ra^{-1}(aK_X + E) + A + N))$$

is globally generated over Y and locally free over S . We can take such A because of Lemma 4.1. For simplicity, we set $\mathcal{G}(l, m) := f_*\mathcal{O}_X(la^{-1}(aK_{X/Y} + E) + mA)$ for every $l, m \in \mathbb{Z}$ with $a|l$. Let $m_0 > 0$ be an integer such that $m_0A - ra^{-1}(aK_X + E)$ is nef for each $0 \leq r < d$ with $a|r$. Let q_e and r_e be the quotient and the remainder of the division of $(p^e - 1)$ by d respectively. Set $D' := D + K_Y$. Then by the assumption and the choice of A ,

$$\begin{aligned} & (\mathcal{G}(1 - p^e, m))(-q_e d D') \\ & \cong f_*\mathcal{O}_X(A - q_e da^{-1}(aK_X + E + af^*D) + (m - 1)A - r_e a^{-1}(aK_{X/Y} + E)) \end{aligned}$$

is globally generated for every $e > 0$ with $a|(p^e - 1)$ and for every $m \geq m_0 + 1$. This implies that $q_e d \leq t_S(\mathcal{G}(1 - p^e, m), D')$. Replacing A if necessary, by Lemma 2.4, we may assume that the morphism

$$f_{Y^e}^* \phi_{(X/Y, E/a)}^{(e)} \otimes \mathcal{O}_{X_{Y^e}}(A_{Y^e}) : \mathcal{G}(1 - p^e, p^e) \rightarrow f_{Y^e}^* \mathcal{O}_{X_{Y^e}}(A_{Y^e}) \cong F_Y^{e*}(\mathcal{G}(0, 1))$$

is surjective over S for every $e > 0$ with $a|(p^e - 1)$. By Keeler's relative Fujita vanishing, there exists $m_1 \geq m_0 + 1$ such that for every f_S -nef Cartier divisor N' on X_S , $\mathcal{O}_{X_S}(m_1 A_S + N')$ is 0-regular with respect to A_S and f_S [15, Example 1.8.24]. Since $-(aK_{X/Y} + E)_S$ is f_S -nef, it follows that the natural morphism

$$\mathcal{G}(0, 1)^{\otimes p^e - m_1} \otimes \mathcal{G}(1 - p^e, m_1) \rightarrow \mathcal{G}(1 - p^e, p^e)$$

is surjective over S for every $e > 0$ with $a|(p^e - 1)$. Thus, by composition, we have a morphism

$$\mathcal{G}(0, 1)^{\otimes p^e - m_1} \otimes \mathcal{G}(1 - p^e, m_1) \rightarrow F_Y^{e*}(\mathcal{G}(0, 1))$$

which is surjective over S . Hence by Lemma 3.5 (1)–(3),

$$(p^e - m_1)t_S(\mathcal{G}(0, 1), D') + t_S(\mathcal{G}(1 - p^e, m_1), D') \leq p^e t_S(\mathcal{G}(0, 1), D'). \quad (4.2.1)$$

Note that since $\mathcal{G}(0, 1)$ is globally generated, $t_S(\mathcal{G}(0, 1), D') \neq -\infty$. In order to get a contradiction, we assume $\mathcal{O}_Y(-bD') \cong \mathcal{O}_Y(-b(K_Y + D))$ is not weakly positive over S . Then by Lemma 3.5 (4) we have $t_S(\mathcal{G}(0, 1), D') \in \mathbb{R}$, and thus

$$q_e d \leq t_S(\mathcal{G}(1 - p^e, m_1), D') \leq m_1 t_S(\mathcal{G}(0, 1), D').$$

This is a contradiction, since $q_e d \xrightarrow{e \rightarrow +\infty} +\infty$.

Next we show (2). By the assumption Δ is \mathbb{Q} -Cartier. Let $m, c \geq 0$ be integers such that $a = mp^c$ and $p \nmid m$. Set $a' := (p^e + 1)m^{-1}$ and $\Delta' := (p^{e-c}E) \otimes a'^{-1}$ for an $e \gg 0$. Then we have $p \nmid a'$ and $\Delta - \Delta' = (p^e + 1)^{-1}\Delta \geq 0$. Thus $(X_{\bar{y}}, \Delta'_{\bar{y}})$ is F -pure for every $y \in S$ and $-(K_X + \Delta' + f^*D)$ is an ample \mathbb{Q} -Cartier divisor. Then $-(K_X + \Delta' + f^*(D' + \varepsilon H))$ is nef for an ample Cartier divisor H on Y and an $\varepsilon \in \mathbb{Q}$ with $0 < \varepsilon \ll 1$. By (1) we see that $\mathcal{O}_Y(-b'(K_Y + D + \varepsilon H))$ is weakly positive over an open subset $Y_1 \subseteq Y$ containing S for a $b' > 0$ divisible enough, and hence $\mathcal{O}_Y(-b(K_Y + D))$ is big over Y_1 . \square

Proof of Theorem 4.3. We use the same notation as the proof of Theorem 4.2 (1). We may assume that Y is smooth and f is flat as above. By the assumption, there exists an open subset $Y_1 \subseteq Y$ containing S such that (i)–(iv) above are satisfied for every $y \in Y_1$, that $\mathcal{O}_X(-b(aK_X + E))$ is f_{Y_1} -free, and that $f_*\mathcal{O}_X(-b(aK_X + E + af^*D))$ is globally generated over Y_1 . Thus we may assume that S is open. Let A be an f -ample and f -free Cartier divisor on X such that $\mathcal{G}(0, 1) = f_*\mathcal{O}_X(A)$ is globally generated and that $f_{Y^e}*\phi_{(X/Y, E/a)}^{(e)} \otimes \mathcal{O}_{X_Y^e}(A_{Y^e})$ is surjective over S for every $e > 0$ with $a|(p^e - 1)$. Then by an argument similar to the above, we obtain the inequality which is the same as (4.2.1). Thus it is enough to show that

$$t_S(\mathcal{G}(1 - p^e, m_1), D') \xrightarrow{e \rightarrow +\infty} +\infty$$

for every $m_1 > 0$, where $D' := D + K_Y$. By the assumption,

$$(\mathcal{G}(-ab, 0))(-abD') \cong f_*\mathcal{O}_X(-b(aK_X + E + af^*D))$$

is globally generated over S , and thus $t_S(\mathcal{G}(-ab, 0), D') \geq ab > 0$. Furthermore, there exists $m_2 > 0$ such that for every $e > 0$ with $a|(p^e - 1)$,

$$\mathcal{G}(-ab, 0)^{\otimes q_e - m_2} \otimes \mathcal{G}(-abm_2 + r_e, m_1) \rightarrow \mathcal{G}(1 - p^e, m_1)$$

is surjective over S , where q_e and r_e are the quotient and the remainder of the division of $(p^e - 1)$ by ab respectively. This implies that

$$c + (q_e - m_2)t_S(\mathcal{G}(-ab, 0), D') \leq t_S(\mathcal{G}(1 - p^e, m_1), D'),$$

where $c := \min\{t_S(\mathcal{G}(-abm_2 - r, m_1), D') | 0 \leq r < ab \text{ and } a|r\}$, and hence the proof is complete. \square

In the remaining part of this section, we will give some corollaries of the above theorems under the following situations. Let $f : X \rightarrow Y$ be a separable surjective morphism between smooth projective varieties, and let Δ be an effective \mathbb{Q} -divisor on X such that $a\Delta$ is Cartier for some $a > 0$. Let D be a \mathbb{Q} -divisor on Y . Let $\bar{\eta}$ be the geometric generic point of Y .

Corollary 4.4. *Assume that f is flat, the support of Δ does not contain any component of any fiber, and (X_y, Δ_y) is F -pure for every closed point $y \in Y$.*

- (1) *If $p \nmid a$ and if $-(K_X + \Delta + f^*D)$ is nef, then so is $-(K_Y + D)$.*
- (2) *If $-(K_X + \Delta + f^*D)$ is ample, then so is $-(K_Y + D)$.*

Proof. This follows from Theorem 4.2 and Remark 3.3 immediately. \square

The author was taught the proof of Corollary 4.5 (3) below by professor Yoshinori Gongyo.

Corollary 4.5. *Assume that $(X_{\bar{\eta}}, \Delta_{\bar{\eta}})$ is F -pure.*

- (1) *If $p \nmid a$ and if $-(K_X + \Delta + f^*D)$ is nef, then $-(K_Y + D)$ is pseudo-effective.*
- (2) *If $-(K_X + \Delta + f^*D)$ is ample, then $-(K_Y + D)$ is big.*
- (3) *If $(X_{\bar{\eta}}, \Delta_{\bar{\eta}})$ is strongly F -regular and if $-(K_X + \Delta + f^*D)$ is nef and big, then $-(K_Y + D)$ is big.*

Proof. By remark 3.3, (1) and (2) of the corollary follow from (1) and (2) of Theorem 4.2 respectively. We prove (3). By Kodaira's lemma, there exists a \mathbb{Q} -divisor $\Delta' \geq \Delta$ on X such that $-(K_X + \Delta' + f^*D)$ is ample and $(X_{\bar{\eta}}, \Delta'_{\bar{\eta}})$ is again strongly F -regular. Hence (2) shows (3). \square

Corollary 4.6. *Assume that $(X_{\bar{\eta}}, \Delta_{\bar{\eta}})$ is F -pure. If $p \nmid a$ and if $K_X + \Delta \sim_{\mathbb{Q}} f^*(K_Y + L)$ for some \mathbb{Q} -Cartier divisor L on Y , then L is pseudo-effective.*

Proof. This follows from Corollary 4.5 (1) by setting $D := -(K_Y + L)$. \square

Remark 4.7. In the same situation as the above, it is known that if $(X_{\bar{\eta}}, \Delta_{\bar{\eta}})$ is globally F -split, then L is \mathbb{Q} -linearly equivalent to an effective \mathbb{Q} -divisor on Y (see [4, Theorem B] or [6, Theorem 3.18]). However, $(X_{\bar{\eta}}, \Delta_{\bar{\eta}})$ is not necessary globally F -split even if $X_{\bar{\eta}}$ is a smooth curve and $\Delta = 0$. Incidentally, Chen and Zhang proved that relative canonical divisors of elliptic fibrations are \mathbb{Q} -linearly equivalent to an effective \mathbb{Q} -divisor on X [3, 3.2].

Remark 4.8. In the case when $\dim Y = 1$, Corollary 4.6 follows from a result of Patakfalvi [19, Theorem 1.6].

Corollary 4.9. *Assume that f is flat and every closed fiber is F -pure.*

- (1) *If X is Fano variety, that is, $-K_X$ is ample, then so is Y .*
- (2) *If the geometric generic fiber of f is strongly F -regular and if X is weak Fano variety, that is, $-K_Y$ is nef and big, then so is Y .*

Proof. This follows from Corollaries 4.4 and 4.5 (3) by setting $D := 0$. \square

Corollary 4.10. *Assume that Y is not a point.*

- (1) *If $(X_{\bar{\eta}}, \Delta_{\bar{\eta}})$ is F -pure, then $-(K_{X/Y} + \Delta)$ is not ample.*
- (2) *If $(X_{\bar{\eta}}, \Delta_{\bar{\eta}})$ is strongly F -regular, then $-(K_{X/Y} + \Delta)$ is not nef and big.*

Proof. This follows from Corollary 4.5 (2) and (3) by setting $D := -K_Y$. \square

Corollary 4.11. *Assume that $\mathcal{O}_X(-m(K_{X/Y} + \Delta))|_{X_{\bar{\eta}}}$ is globally generated for some $m > 0$ with $a|m$. If $p \nmid a$ and if $(X_{\bar{\eta}}, \Delta_{\bar{\eta}})$ is F -pure, then $f_*\mathcal{O}_X(-m(K_{X/Y} + \Delta))$ is not big.*

Proof. Set $\mathcal{G}(n) := f_*\mathcal{O}_X(n(K_{X/Y} + \Delta))$ for every $n \in \mathbb{Z}$ with $a|n$. In order to get a contradiction, we assume that $\mathcal{G}(-m)$ is big. Then $(S^l\mathcal{G}(-m))(-H)$ is globally generated over the generic point $\eta \in Y$ for some $l > 0$ and some ample divisor H on Y . By the assumption, there exists an $n_0 > 0$ such that the natural morphism

$$(S^l\mathcal{G}(-m))^{\otimes n-n_0} \otimes \mathcal{G}(-lmn_0) \rightarrow \mathcal{G}(-lmn)$$

is generically surjective for every $n \geq n_0$. Let $n_1 > 0$ be an integer such that $(\mathcal{G}(-lmn_0))(n_1H)$ is globally generated. Set $H_n := (n - n_0 - n_1)(lmn)^{-1}H$. Then

we see that

$$(\mathcal{G}(-lmn))((n_0 + n_1 - n)H) \cong f_*\mathcal{O}_X(-lmn(K_{X/Y} + \Delta + H_n))$$

is globally generated over η . Applying Theorem 4.3, we get $-H_n$ is pseudo-effective for every $n \geq n_0$. This is a contradiction. \square

Remark 4.12. We cannot remove the assumption of F -purity of fibers in Corollaries 4.6 and 4.11. Indeed, there is a quasi-elliptic fibration $g : S \rightarrow C$, that is, a surjective morphism from a smooth projective surface S to a smooth projective curve C whose general fibers are cuspidal curve of arithmetic genus one, such that $K_{S/C} \sim_{\mathbb{Q}} g^*L$ for a \mathbb{Q} -divisor L on C with $\deg L < 0$ [23][27, Theorem 3.6]. Note that cuspidal singularities are not F -pure [9].

5. RESULTS IN ARBITRARY CHARACTERISTIC

In this section we generalize some results in Section 4 to arbitrary characteristic by reduction to positive characteristic. In particular, we prove characteristic zero counterparts of Corollaries 4.10 and 4.11 (Theorems 5.4 and 5.5). In the last of this section, we deal with morphisms which are special but not necessarily smooth, and show that images of Fano varieties are again Fano varieties. First we recall some classes of singularities in characteristic zero defined by reduction to positive characteristic.

Definition 5.1. Let X be a normal variety over a field k of characteristic zero, and let Δ be an effective \mathbb{Q} -Weil divisor on X . Let (X_R, Δ_R) be a model of (X, Δ) over a finitely generated \mathbb{Z} -subalgebra R of k . (X, Δ) is said to be of *dense F -pure type* (resp. *strongly F -regular type*) if there exists a dense (resp. dense open) subset $S \subseteq \operatorname{Spec} R$ such that (X_μ, Δ_μ) is F -pure (resp. strongly F -regular) for all closed points $\mu \in S$.

Remark 5.2. The above definition can be generalized to the case where X is a finite disjoint union of varieties over k obviously.

Theorem 5.3 ([24, Corollary 3.4]). *Let X be a normal variety over a field of characteristic zero, and let Δ be an effective \mathbb{Q} -Weil divisor on X such that $K_X + \Delta$ is \mathbb{Q} -Cartier. Then (X, Δ) is klt if and only if it is of strongly F -regular type.*

Let $f : X \rightarrow Y$ be a surjective morphism between smooth projective varieties over an algebraically closed field of characteristic zero, and let Δ be an effective \mathbb{Q} -divisor on X . Assume that the dimension of Y is positive.

Theorem 5.4. *If (X_y, Δ_y) is of dense F -pure type (resp. klt) for every general closed point $y \in Y$, then $-(K_{X/Y} + \Delta)$ is not ample (resp. not nef and big). In particular, $-K_{X/Y}$ is not nef and big.*

Proof. Assume that (X_y, Δ_y) is of dense F -pure type for a general closed point $y \in Y$. Let X_R, Δ_R, Y_R, y_R and f_R be respectively models of X, Δ, Y, y and f over a finitely generated \mathbb{Z} -algebra R . We may assume that $(X_R)_{y_R}$ is a model of X_y over R . Then there exists a dense subset $S \subseteq \operatorname{Spec} R$ such that $((X_y)_\mu, \Delta_\mu)$ is F -pure for every $\mu \in S$. Note that $(X_y)_\mu \cong (X_\mu)_{y_\mu}$ and $(\Delta_y)_\mu = (\Delta_\mu)_{y_\mu}$. Thus by Corollary 4.10, we see that $-(K_{X_\mu/Y_\mu} + \Delta_\mu)$ is not ample. This implies that $-(K_{X/Y} + \Delta)$

is not ample. Next, we assume that (X_y, Δ_y) is klt for every general closed point $y \in Y$. If $-(K_{X/Y} + \Delta)$ is nef and big, then by Kodaira's lemma, there exists $\Delta' \geq \Delta$ such that (X_y, Δ'_y) is klt for a general point $y \in Y$ and $-(K_{X/Y} + \Delta')$ is ample. However, by Theorem 5.3, (X_y, Δ'_y) is of dense F -pure type, which contradicts to the above arguments. \square

Theorem 5.5. *Assume that (X_y, Δ_y) is of dense F -pure type for a general point $y \in Y$. Let $\bar{\eta}$ be a geometric generic point of Y . If $\mathcal{O}_X(-m(K_{X/Y} + \Delta))|_{X_{\bar{\eta}}}$ is globally generated for some $m > 0$ such that $m\Delta$ is integral, then $f_*\mathcal{O}_X(-m(K_{X/Y} + \Delta))$ is not big.*

Proof. Set $\mathcal{G} := f_*\mathcal{O}_X(-m(K_{X/Y} + \Delta))$ and $r := \text{rank } \mathcal{G}$. Since $y \in Y$ is general, f is flat over y and $\dim H^0(X_y, -m(K_{X_y} + \Delta_y)) = r$. Let X_R, Δ_R, Y_R, y_R and f_R be respectively models of X, Δ, Y, y and f . By replacing R if necessary, we may assume that $f_{R*}\mathcal{O}_{X_R}(-m(K_{X_R/Y_R} + \Delta_R))$ and $(X_R)_{y_R}$ are models of \mathcal{G} and X_y respectively, and that for every $\mu \in \text{Spec } R$, $\dim H^0((X_\mu)_{y_\mu}, -m(K_{X_\mu} + \Delta_\mu)_{y_\mu}) = r$. Hence by [11, Corollary 12.9], the natural morphism

$$\mathcal{G}_\mu = f_{R*}\mathcal{O}_{X_R}(-m(K_{X_R/Y_R} + \Delta_R))|_{Y_\mu} \rightarrow f_{\mu*}\mathcal{O}_{X_\mu}(-m(K_{X_\mu/Y_\mu} + \Delta_\mu))$$

is surjective over y_μ . Since $f_{\mu*}\mathcal{O}_{X_\mu}(-m(K_{X_\mu/Y_\mu} + \Delta_\mu))$ is not big by Corollary 4.11, \mathcal{G}_μ is also not big. Thus the lemma below completes the proof. \square

Lemma 5.6. *Let \mathcal{G} be a torsion free coherent sheaf on a smooth projective variety Y over an algebraically closed field of characteristic zero. Let Y_R and \mathcal{G}_R be models of Y and \mathcal{G} respectively over a finitely generated \mathbb{Z} -algebra. If \mathcal{G} is big, then there exists a dense open subset $S \subseteq \text{Spec } R$ such that \mathcal{G}_μ is big for every $\mu \in S$.*

Proof. Replacing Y by its appropriate open subset, we may assume that \mathcal{G} is locally free. Then there exists a morphism $\varphi : \bigoplus \mathcal{O}_Y(H) \rightarrow S^a \mathcal{G}$ for an ample divisor H on Y and an $a > 0$, which is surjective over a closed point $y \in Y$. Let φ_R, H_R and y_R be models of φ, H and y over R respectively. By replacing R if necessary, we may assume that φ_R is surjective over y_R . Thus for every closed point $\mu \in \text{Spec } R$, the morphism $\varphi_\mu : \bigoplus \mathcal{O}_{X_\mu}(H_\mu) \rightarrow \mathcal{G}_\mu$ obtained as the reduction of φ_R is surjective over y_μ . This implies that \mathcal{G}_μ is big, since H_μ is ample. \square

Kollár, Miyaoka and Mori [14, Corollary 2.9] (cf. [17, THEOREM 3]) proved that images of Fano varieties under smooth morphisms are again Fano varieties. The next theorem shows that the same statement holds when morphisms are not necessary smooth.

Theorem 5.7. *Let $f : X \rightarrow Y$ be a surjective morphism between smooth projective varieties over an algebraically closed field k of any characteristic $p \geq 0$, and let Δ be an effective \mathbb{Q} -divisor on X such that $a\Delta$ is integral for some $0 < a \in \mathbb{Z} \setminus p\mathbb{Z}$. Assume that for every closed point $x \in X$, there exist a neighborhood $U \subseteq X$ (resp. $V \subseteq Y$) of x (resp. $f(x)$) and a commutative diagram*

$$\begin{array}{ccc} U & \xrightarrow{\alpha} & \mathbb{A}^m = \text{Spec } k[u_1, \dots, u_m] \\ (f_V)|_U \downarrow & & \downarrow \varphi \\ V & \xrightarrow{\beta} & \mathbb{A}^n = \text{Spec } k[v_1, \dots, v_n] \end{array}$$

whose horizontal morphisms are étale, that the morphism φ is defined as

$$\varphi(a_1, \dots, a_m) = \left(\prod_{0 < i \leq l_1} a_i, \prod_{l_1 < i \leq l_2} a_i, \dots, \prod_{l_{n-1} < i \leq l_n} a_i \right) \text{ with } 0 < l_1 < \dots < l_n \leq m,$$

and that

$$\Delta|_U = \alpha^* \left(\sum_{l_n < i \leq m} d_i \operatorname{div}(u_i) \right) \text{ with } d_{l_n+1}, \dots, d_m \in \mathbb{Z}_{(p)} \cap [0, 1].$$

In this situation, if $-(K_X + \Delta + f^*D)$ is ample for some \mathbb{Q} -Cartier divisor D on Y , then so is $-(K_Y + D)$.

Proof. When the characteristic of k is zero, it is easily seen that the entire setting can be reduced to characteristic $p \gg 0$. Thus we only need to consider the case when $p > 0$. Let $x \in X$ be a closed point, and let U, V, α, β and φ be as above. Set $y := f(x)$, $\mathbf{a} := \alpha(x) = (a_1, \dots, a_m) \in \mathbb{A}^m$ and $\mathbf{b} := \beta(y) = (b_1, \dots, b_n) \in \mathbb{A}^n$. Set $u'_i := u_i - a_i$ and $v'_j := v_j - b_j$ for each i and j . Then

$$\varphi^* v'_j = \prod_{l_{j-1} < i \leq l_j} (u'_i + a_i) - \prod_{l_{j-1} < i \leq l_j} a_i$$

for $j = 1, \dots, n$, where $l_0 := 0$. Set $g := \prod_{l_n < i \leq m} u_i$. It is easy to check that the sequence $\varphi^* v'_1, \dots, \varphi^* v'_n, g$ is $k[u_1, \dots, u_m]$ -regular. In particular, $Z := Z(\varphi^* v'_1, \dots, \varphi^* v'_n) \subseteq \mathbb{A}^m$ is equi-dimensional of dimension $m - n$. Replacing V and U if necessary, we may assume that $\beta^{-1}(\beta(y)) = \{y\}$. Then we obtain the étale morphism $\alpha_{\mathbf{b}} : U_y \rightarrow Z$. This implies that every closed fiber of f is equi-dimensional, in particular f is flat.

Claim. (X_y, Δ_y) is F -pure for every closed point $y \in Y$.

If this claim holds, then the theorem follows from Corollary 4.4, because by the assumption the support of Δ does not contain any component of any fiber. Since $\sum_{l_n < i \leq m} d_i \operatorname{div}(u_i) \leq \operatorname{div}(g)$, it suffice to show that the pair $(Z, \operatorname{div}(g)|_Z)$ is F -pure around \mathbf{a} . Let $\mathfrak{m}_{\mathbf{a}}$ be the maximal ideal of \mathbf{a} . Then it is easily seen that

$$(\varphi^* v'_1 \cdots \varphi^* v'_n)^{q-1} \cdot g^{q-1} \notin \mathfrak{m}_{\mathbf{a}}^{[q]}$$

for every $q = p^e$. Thus by [10, Corollary 2.7], the pair $(Z, \operatorname{div}(g)|_Z)$ is F -pure, which completes the proof. \square

6. QUESTIONS

The first question is on a characteristic zero counterpart of Theorem 4.2. Let $f : X \rightarrow Y$ be a surjective morphism between smooth projective varieties over an algebraically closed field of characteristic zero. Since F -pure singularities are considered to correspond to semi-log canonical singularities (cf. [16]), we consider the case where every closed fiber is semi-log canonical.

Question 6.1. Assume that $-K_X$ is ample (resp. nef, nef and big). Is $-K_Y$ also ample (resp. nef, nef and big)?

The next question is on a generalization of Theorem 1.6. Let $f : X \rightarrow Y$ be a generically smooth surjective morphism between smooth projective varieties over an algebraically closed field. Assume that $-K_{X/Y}$ is nef. By Theorem 1.6, we see that if the Iitaka-Kodaira dimension $\kappa(X, -K_{X/Y})$ of $-K_{X/Y}$ is equal to $\dim X$, then $\dim Y = 0$. In other words, if $\kappa(X, -K_{X/Y})$ is the maximum, then $\dim Y$ is the minimum. It is expected that this phenomenon is also obtained as a consequence of a more general phenomenon. The following inequality is one of possibilities.

Question 6.2. Does the inequality $\kappa(X, -K_{X/Y}) \leq \kappa(X_{\overline{\eta}}, -K_{X_{\overline{\eta}}})$ hold?

If the inequality holds, then the above phenomenon can be viewed as its consequence. For instance, in characteristic zero, Question 6.2 is known in the case where $X = \mathbb{P}(\mathcal{E})$ for a vector bundle \mathcal{E} of rank r on Y and f is the natural projection. In this case, if $-K_{X/Y}$ is nef, then the numerical dimension of $-K_{X/Y}$ is equal to $r - 1$ [18, 4.7. Corollary] (cf. [26]). Thus

$$\kappa(X, -K_{X/Y}) \leq r - 1 = \dim X_{\overline{\eta}} = \kappa(X_{\overline{\eta}}, -K_{X_{\overline{\eta}}}).$$

REFERENCES

- [1] C. Birkar, Y. Chen: *Images of manifolds with semi-ample anti-canonical divisor*, to appear in J. Algebraic Geom.
- [2] M. Chen, Q. Zhang: *On a question of Demailly-Peternell-Schneider*, J. Eur. Math. Soc. **15**, 1853–1858 (2013).
- [3] Y. Chen, L. Zhang: *The subadditivity of the Kodaira dimension for fibrations of relative dimension one in positive characteristics*, to appear in Math. Res. Lett. (2013).
- [4] O. Das, K. Schwede: *The F -different and a canonical bundle formula*, <http://arxiv.org/abs/1508.07295> (2015).
- [5] O. Debarre: *Higher-Dimensional Algebraic Geometry*, Universitext, SpringerVerlag, New York, (2001).
- [6] S. Ejiri: *Weak positivity theorem and Frobenius stable canonical rings of geometric generic fibers*, <http://arxiv.org/abs/1508.00484> (2015).
- [7] O. Fujino, Y. Gongyo: *On images of weak Fano manifolds*, Math. Z. **270** (2012), no 1, 531–544.
- [8] O. Fujino, Y. Gongyo: *On images of weak Fano manifolds II*, Algebraic and Complex Geometry, Springer Proceedings in Mathematics & Statistics, **71** (2014), 201–207.
- [9] S. Goto, K. Watanabe: *The structure of one-dimensional F -pure rings*, J. Algebra **49** (1977), 415–421.
- [10] N. Hara, K.-i. Watanabe: *F -regular and F -pure rings vs. log terminal and log canonical singularities*, J. Algebraic Geom. **11** (2002), no. 2, 363–392.
- [11] R. Hartshorne: *Algebraic geometry*, Grad. Texts in Math. no **52**, Springer-Verlag, NewYork, (1977).
- [12] D. S. Keeler: *Ample Filters of invertible sheaves*, J. Algebra **259** (2003), 243–283.
- [13] J. Kollár: *Subadditivity of the Kodaira dimension: fibers of general type*, Algebraic geometry, Sendai, 1985, 361–398, Adv. Stud. Pure Math., **10**, North-Holland, Amsterdam, (1987).

- [14] J. Kollár, Y. Miyaoka, S. Mori: *Rational connectedness and boundedness of Fano manifolds*, J. Differential Geom. **36** (1992), no 3, 765–779.
- [15] R. Lazarsfeld: *Positivity in algebraic geometry II*, Ergebnisse der Mathematik und ihrer Grenzgebiete, 3. Folge, **49** Springer-Verlag, Berlin, (2004).
- [16] L. E. Miller, K. Schwede: *Semi-log canonical vs F -pure singularities*, J. Algebra **349** (2012), 150–164.
- [17] Y. Miyaoka: *Relative deformations of morphisms and applications to fibre spaces*, Comment. Math. Univ. St. Paul. **42** (1993), no 1, 1–7.
- [18] N. Nakayama, *Zariski decomposition and abundance*, MSJ Memoirs, **14**. Mathematical Society of Japan, Tokyo, (2004).
- [19] Z. Patakfalvi: *Semi-positivity in positive characteristics*, Ann. Sci. Ecole Norm. S. **47** (2014), no. 5, 991–1025.
- [20] Z. Patakfalvi: *On subadditivity of Kodaira dimension in positive characteristic*, <http://arxiv.org/abs/1308.5371> (2013).
- [21] Z. Patakfalvi, K. Schwede, W. Zhang: *F -singularities in families*, <http://arxiv.org/abs/1305.1646> (2013).
- [22] Yu. G. Prokhorov, V. V. Shokurov: *Towards the second main theorem on complements*, J. Algebraic Geom. **18** (2009), 151–199.
- [23] M. Raynaud: *Contre-exemple au "vanishing theorem" en caractéristique $p > 0$* , C.P.Ramanujam –A tribute, Studies in Math. **8** (1978), 273–278.
- [24] S. Takagi: *An interpretation of multiplier ideals via tight closure*, J. Algebraic Geom. **13** (2004), 393–415.
- [25] E. Viehweg: *Quasi-projective moduli for polarized manifolds*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) **30** Springer-Verlag, Berlin, (1995).
- [26] K. Yasutake: *On projective space bundles with nef normalized tautological divisor*, <http://arxiv.org/abs/1104.5084> (2011).
- [27] Q. Xie: *Counterexamples to the Kawamata-Viehweg vanishing on ruled surfaces in positive characteristic*, J. Algebra **324** (2010), no. 12, 3494–3506.

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